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Some measurements of multi-point time correlations in grid turbulence

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Three-point odd-order correlations and four-point even-order correlations of the longitudinal velocity fluctuations in grid-generated turbulence have been measured using linearized hot-wire anemometry, digital sampling, and a high-speed digital computer. The measured correlations are compared with relations between higher-order correlations corresponding to non-Gaussian Gram-Charlier joint probability densities for three and four variables. The fourth-order, three-point Gram-Charlier distribution accurately describes the relation between measured odd-order three-point correlations. The measured fourth-order even-order correlations may be accurately predicted from the two-point correlation using Millionshtchikov's joint-Gaussian hypothesis, except for small values of the separations. The disagreement at small separations cannot be reduced through use of the Gram-Charlier approximation.

1. Introduction

Recent work by Frenkiel & Klebanoff (1967*a*) and Van Atta & Chen (1968, hereinafter often referred to as I) has experimentally determined the relations between two-point higher-order time correlations for grid-generated turbulence. These studies showed that except for very small values of the time separation all two-point even-order correlations may be accurately predicted from the second-order correlation by assuming a Gaussian joint probability density for fluctuations at different times, while a non-Gaussian Gram-Charlier distribution describes relations between the odd-order correlations remarkably well. These findings involve two-point correlations only, whereas some theoretical investigations are concerned with correlations defined at a larger number of points and with the relations between associated higher and lower order correlations. In the present investigation, we have therefore extended the previous measurements to some three- and four-point correlations and compared the measured correlations with corresponding results for Gaussian and non-Gaussian probability distributions in the hope that the results may prove useful for future theoretical investigations.

2. Experimental arrangement

The experiments were carried out in the 76 cm by 76 cm by 9 m test section of the low-turbulence wind tunnel in the Department of Aerospace and Mechanical

Engineering Sciences. A biplane grid of round, polished dural rods was located 2·4 m from the end of the contraction section. The grid mesh size M was 5·08 cm with rods of 0·953 cm diameter. The mean velocity V was 7·7 m/sec, and the corresponding Reynolds number based on mesh spacing was 25,300. All detailed digital measurements were made at $x/M = 48$, where x is the distance downstream from the grid.

A tungsten hot-wire, 1 mm long and 5μ diameter was used to measure u, the longitudinal fluctuating component of the turbulent velocity field. A DISA 55A01 amplifier was used to operate the hot-wire at constant resistance with overheat ratio of 0·5. The hot-wire output was linearized using a DISA 55D10 linearizer. The linearized hot-wire signal was FM tape recorded at a tape speed of 152·4 cm/sec using a Sanborn 3917 A recorder. The analogue tape was later played back and sampled with an analogue-to-digital converter at a rate of 5616 samples per second, somewhat faster than twice the highest frequency for which the turbulent spectrum was unmistakably distinguishable from electronic noise. These digital data were in fact a part of the data available from an earlier study of two-point correlations by Van Atta & Chen (1968) and the present measurements therefore directly complement and extend these earlier results. Using a CDC 3600 computer, the digital data were processed in several sections, using an averaging time of 54·7 sec (307,200 digital velocity samples) for each section. All even-order correlations computed were based on four samples of this length, while all odd-order correlations were based on seven samples of this length.

3. Computation method

In the present study, we consider time correlations derived from a knowledge of the time history of the velocity at a single point. If one invokes Taylor's hypothesis, the time correlations may be alternately interpreted as space correlations in a homogeneous turbulent field. Thus, an n-point time correlation may be reinterpreted as an n-point space correlation.

Some three-point time correlations have previously been computed in unpublished measurements by Frenkiel & Klebanoff (1967b) using direct computation of mean lagged products. We have not used this method in the present work, but have instead employed a slight modification of the fast-Fourier transform method described by Van Atta & Chen (1968). Before describing the computation method actually used, it may be of value to point out a similar but more general method of computing nth order correlations which was not used in the present work, partially because of computer memory limitations.

The $(n+1)$ point correlation is defined as

$$R(\tau_1, \tau_2, \ldots, \tau_n)\langle u^2\rangle^{\frac{1}{2}(n+1)} = \langle u(t)u(t+\tau_1)\ldots u(t+\tau_n)\rangle$$
$$= (1/T)\int_{-\frac{1}{2}T}^{\frac{1}{2}T} u(t)\,u(t+\tau_1)\ldots u(t+\tau_n)\,dt. \qquad (1)$$

If we take the n-dimensional finite *Fourier transform* of the correlation, we obtain the nth order spectrum

$$S_n(f_1, f_2, \ldots, f_n)$$
$$= \int_{-\infty}^{\infty} \ldots \int R(\tau_1, \ldots, \tau_n) \langle u^2 \rangle^{\frac{1}{2}(n+1)} \exp\left(-i2\pi \sum_i f_i \tau_i\right) d\tau_1, \ldots, d\tau_n$$
$$= (1/T) \int_{-\infty}^{\infty} \ldots \int \int_{-\frac{1}{2}T}^{\frac{1}{2}T} [u(t)u(t+\tau_1) \ldots u(t+\tau_n)] \exp\left(-i2\pi \sum_i f_i \tau_i\right) d\tau_1, \ldots, d\tau_n dt. \quad (2)$$

Letting $t_i = t + \tau_i$ ($i = 1, \ldots, n$), (2) becomes

$$S_n(f_1, \ldots, f_n) = (1/T) \int_{-\frac{1}{2}T}^{\frac{1}{2}T} u(t) \exp(i2\pi(\Sigma f_i)t)\, dt \left[\prod_i^n \int_{-\infty}^{\infty} u(t_i) \exp(-i2\pi f_i t_i)\, dt_i\right]$$
$$= (1/T) U[-(f_1 + f_2 + \ldots + f_n)] U(f_1) U(f_2) \ldots U(f_n)$$
$$= (1/T) \left[\prod_i^n U(f_i)\right] U^*(f_1 + f_2 + \ldots + f_n). \quad (3)$$

$U(f)$ is the first-order complex discrete Fourier transform of the signal given by

$$U(f) = \sum_{t=0}^{N-1} u_t \exp(-i2\pi ft/N) \quad (f = 0, 1, \ldots, N-1),$$

which is equivalent to the transform of the truncated continuous function

$$U(f) = \int_{-\frac{1}{2}T}^{\frac{1}{2}T} u(t) \exp(-i2\pi ft)\, dt.$$

From (3) we see that the nth order spectrum is determined simply as a product of the values of the one-dimensional complex Fourier transform at certain combinations of frequencies, and hence all higher order spectra may be determined from a single transform of the time series. The symmetries associated with such polyspectra and their interpretation have been discussed by Brillinger & Rosenblatt (1967).

The $(n+1)$-point correlation may then be obtained by taking the n-dimensional inverse Fourier transform of S_n

$$R(\tau_1, \tau_2, \ldots, \tau_n) \langle u^2 \rangle^{\frac{1}{2}(n+1)} = \int_{-\infty}^{\infty} \ldots \int S_n(f_1, f_2, \ldots, f_n) \exp\left(i2\pi \sum_i^n f_i \tau_i\right) df_1, \ldots, df_n. \quad (4)$$

As explained by Cooley & Tukey (1965), if one uses the fast-Fourier transform algorithm to perform the transforms, it is computationally faster to obtain both the spectrum and the correlation function in this manner than to obtain the correlation function alone by directly computing mean lagged products. Although this method is attractive in principle, one sees from the form of (3) that memory requirements for storing the arrays to be inverse transformed increase sharply with n. In fact, in the present measurements we found it impractical to attempt to use this method even for the case $n = 2$ (bi-spectrum), which requires a two-dimensional Fourier transform. We have therefore computed the higher-order correlations using one-dimensional transforms only, by adopting the following procedure.

We define a modified form of the correlation

$$R(\tau_1, \ldots, \tau_k, \tau, \tau+\tau_1', \ldots, \tau+\tau_l')\langle u^2\rangle^{\frac{1}{2}(k+l+2)} = R(\tau_1, \tau_2, \ldots, \tau_{k+l+1})\langle u^2\rangle^{\frac{1}{2}(k+l+2)}$$
$$= \langle u_k(\tau_1, \tau_2, \ldots, \tau_k, t)\, u_l(\tau_1', \tau_2', \ldots, \tau_l', t+\tau)\rangle, \quad (5)$$

where $$u_k(\tau_1, \tau_2, \ldots, \tau_k, t) \equiv u(t)\,u(t+\tau_1)\ldots u(t+\tau_k). \quad (6)$$

Taking the transform with respect to τ produces the one-dimensional spectrum

$$S_{kl}(\tau_1, \ldots, \tau_k, \tau_1', \ldots, \tau_l', f)$$
$$= \int_{-\infty}^{\infty} R(\tau_1, \ldots, \tau_k, \tau, \tau+\tau_1', \ldots, \tau+\tau_l')\langle u^2\rangle^{\frac{1}{2}(k+l+2)} \exp(-i2\pi f\tau)\,d\tau$$
$$= (1/T)\,[U_l(\tau_1', \tau_2', \ldots, \tau_l', f)\,U_k^*(\tau_1, \tau_2, \ldots, \tau_k, f)], \quad (7)$$

where $$U_k(\tau_1, \tau_2, \ldots, \tau_k, f) = \int_{-\infty}^{\infty} u_k(\tau_1, \tau_2, \ldots, \tau_k, t) \exp(-i2\pi ft)\,dt. \quad (8)$$

The correlation produced by the inverse transform is then

$$R(\tau_1, \ldots, \tau_k, \tau, \tau+\tau_1', \ldots, \tau+\tau_l')\langle u^2\rangle^{\frac{1}{2}(k+l+2)}$$
$$= \int_{-\infty}^{\infty} S_{kl}(\tau_1, \ldots, \tau_k, \tau_1', \ldots, \tau_l', f) \exp(i2\pi f\tau)\,df. \quad (9)$$

The essential new feature is that the product $u_k(\tau_1, \tau_2, \ldots, \tau_k, t)$, where the $\tau_1, \ldots, \tau_k$ are fixed, is treated as a new time series and processed in the usual fashion. The correlation function is obtained for all values of τ, but only for given fixed values of each τ_k. The procedure must be repeated for each new value of τ_k, whereas the general method previously outlined produced the correlation function for all values of the τ_k in one cycle of the computation. This general procedure is exhibited more clearly in the results for particular correlations that follow.

4. Measured correlations

4.1. *Three-point odd-order velocity correlations*

In the following sections, we adopt the superscript notation introduced by Frenkiel & Klebanoff (1967*a*) to explicitly denote the number of points and powers of the longitudinal fluctuating velocity involved in each correlation.

The three-point third-order correlation is

$$R^{1,1,1}(\tau_1, \tau)\langle u^2\rangle^{\frac{3}{2}} = \langle u(t)\,u(t+\tau_1)\,u(t+\tau)\rangle.$$

For this case, we have $$u_1(\tau_1, t) = u(t)\,u(t+\tau_1) \quad (10)$$

and $$U_1(\tau_1, f) = \int_{-\infty}^{\infty} u_1(\tau_1, t) \exp(-i2\pi ft)\,dt. \quad (11)$$

The complex spectrum is

$$S_{1,0}(\tau_1, f) = (1/T)[U(f)\,U_1^*(\tau_1, f)] \quad (12)$$

and the correlation is given by

$$R^{1,1,1}(\tau_1, \tau)\,\langle u^2\rangle^{\frac{3}{2}} = \int_{-\infty}^{\infty} S_{1,0}(\tau_1, f) \exp(i2\pi ft)\,df. \quad (13)$$

The results for the three-point correlation for positive τ and six fixed values of τ_1 are shown in figure 1, and for negative τ and several fixed values of τ_1 in figure 2. The behaviour of the correlation in other quadrants of the variables τ_1 and τ and along certain preferred axes (e.g. $\tau_1 = \tau_2$) can be found from the symmetry properties of the correlation. This correlation has been mentioned by Meecham & Jeng (1968) as being of vital importance in some theoretical studies. Here we

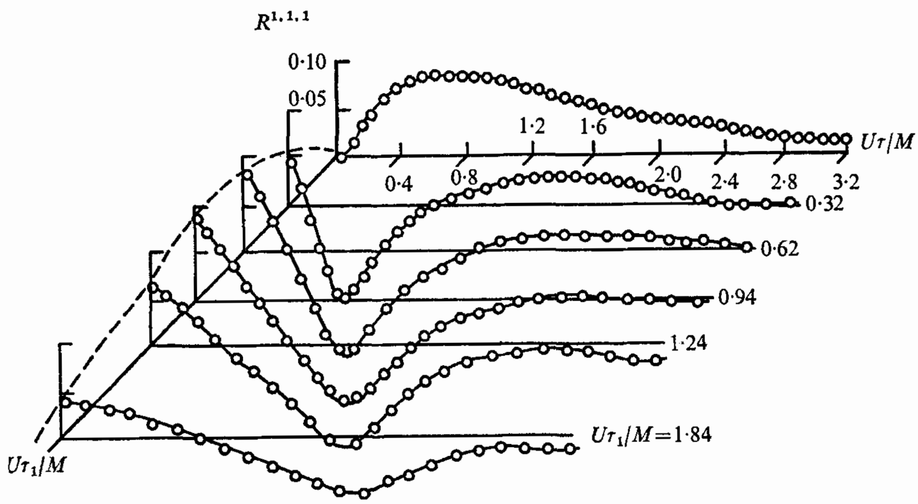

FIGURE 1. Triple correlations. $R^{1,1,1}(\tau_1,\tau) = \langle u(t)u(t+\tau_1)u(t+\tau)\rangle/\langle u^2\rangle^{\frac{3}{2}}$ for fixed values of $U\tau_1/M$.

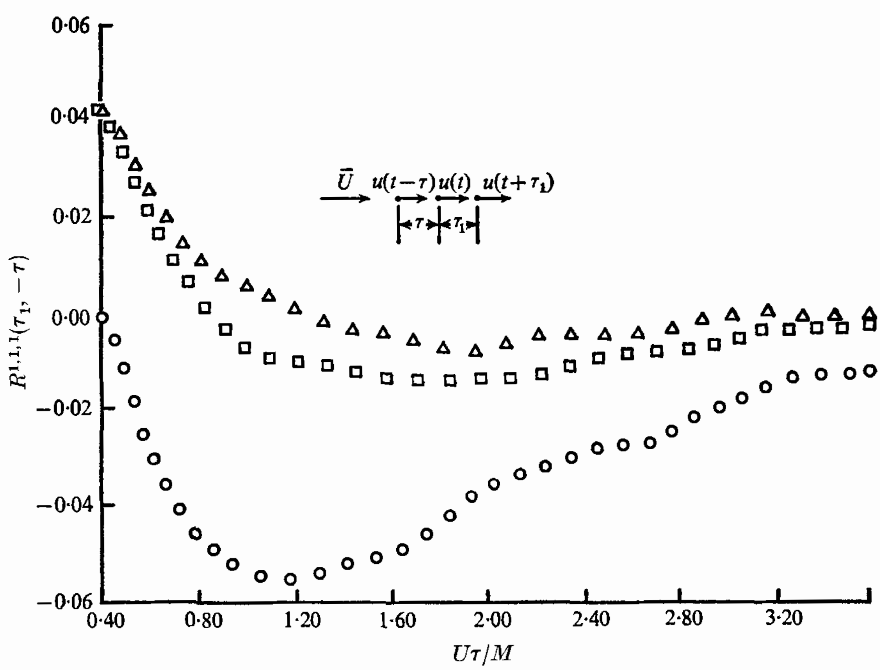

FIGURE 2. Triple correlations. $R^{1,1,1}(\tau_1,\tau) = \langle u(t)u(t+\tau_1)u(t-\tau)\rangle/\langle u^2\rangle^{\frac{3}{2}}$ for fixed values of $U\tau_1/M$: ○, 0·0000; □, 0·6208; △, 0·9446.

shall not attempt to discuss the present data in this context, but present these results because they are needed to compute the fifth-order, three-point correlation using the relations corresponding to the Gram-Charlier distribution in order to compare with the directly measured fifth-order, three-point correlations.

Several fifth-order, three-point correlations were computed in order to determine whether the Gram-Charlier approximation for the three-point joint probability density can successfully describe relations between odd-point correlations

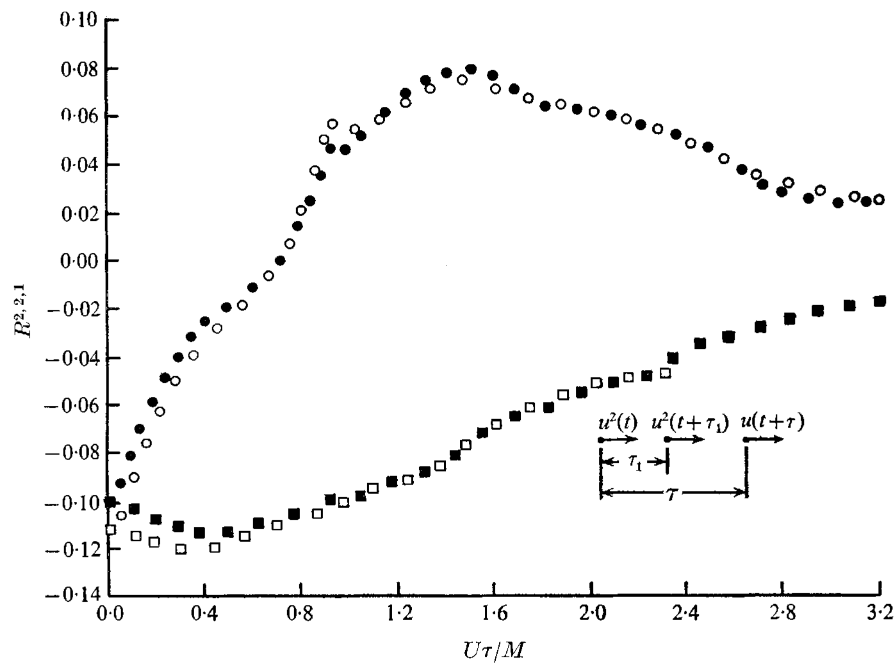

FIGURE 3. Three-point, fifth-order correlations. $R^{2,2,1}(\tau_1,\tau) = \langle u^2(t)u^2(t+\tau_1)u(t+\tau)\rangle/\langle u^2\rangle^{\frac{5}{2}}$ for fixed $U\tau_1/M = 0{\cdot}94464$. (*a*) Measured correlations: ●, positive τ; ■, negative τ. (*b*) Calculated from lower order correlations using Gram-Charlier fourth-order non-Gaussian joint probability distribution: ○, positive τ; □, negative τ.

of different order as was found for odd-order, two-point correlations by Frenkiel & Klebanoff (1967*a*) and in I. The additional functions $U_3(0,\tau_1,\tau_1,f)$ and $U_3(0,0,\tau_1,f)$ were formed and combined with previously available functions to calculate

$$R^{2,2,1}(\tau_1,\tau)\langle u^2\rangle^{\frac{5}{2}} = \langle u^2(t)\,u^2(t+\tau_1)\,u(t+\tau)\rangle$$

and

$$R^{3,1,1}(\tau_1,\tau)\langle u^2\rangle^{\frac{5}{2}} = \langle u^3(t)\,u(t+\tau_1)\,u(t+\tau)\rangle.$$

The measured correlations are shown in figures 3 and 4, and are compared with the corresponding fifth-order, three-point correlations calculated from the measured third-order, three-point correlations using the relations for a fourth-order, three-point Gram-Charlier distribution given by Van Atta & Yeh (1970). The appropriate relations are

$$R^{2,2,1} = 4R_{1,2}R^{1,1,1} + 2(R_{2,3}R^{2,1,0} + R_{1,3}R^{1,2,0}) + R_{2,3} + R^{2,0,1}$$

and

$$R^{3,1,1} = 3(R^{1,1,1} + R_{1,2}R^{2,0,1} + R_{1,3}R^{2,1,0}) + R_{2,3}R^{3,0,0},$$

where a subscript notation has been adopted for the two-point correlations, designating at which of the three points the velocities are measured; e.g. $R_{2,3} = R^{0,1,1}$. There is good agreement between the directly measured and computed correlations, and we conclude that the relations between odd-point correlations corresponding to the Gram-Charlier approximation provide an accurate description of the experimental data. For two-point correlations the approximation works well up to the highest order so far considered (seventh), and it seems likely that the approximation will continue to be accurate for higher order three-point correlations.

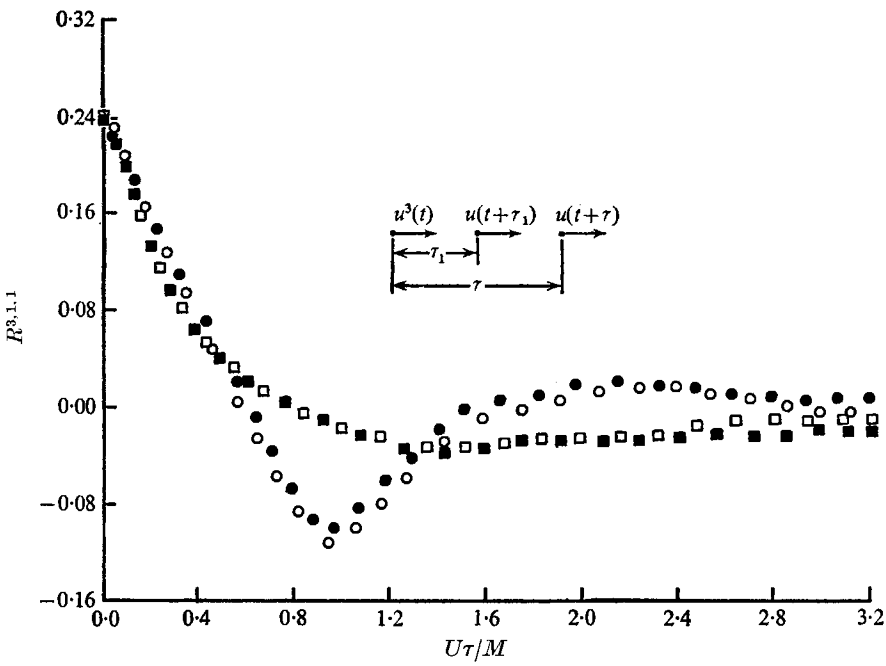

FIGURE 4. Three-point, fifth-order correlations. $R^{3,1,1}(\tau_1, \tau) = \langle u^3(t)\, u(t+\tau_1)\, u(t+\tau)\rangle / \langle u^2\rangle^{\frac{5}{2}}$ for fixed $U\tau_1/M = 0{\cdot}94464$. (*a*) Measured correlations: ●, positive τ; ■, negative τ. (*b*) Calculated from lower order correlations using Gram-Charlier fourth-order non-Gaussian joint probability distribution: ○, positive τ; □, negative τ.

4.2. *Four-point, fourth-order correlations*

The four-point, fourth-order correlation was computed in order to test the accuracy of Millionshtchikov's (1939) hypothesis for four-point time correlations. As originally formulated, this hypothesis provides an expression for two-point fourth-order correlations in terms of second-order correlations under the assumption of a joint-normal probability distribution for the velocities at the two points. The measurements of Frenkiel & Klebanoff (1967*a*) and those reported in I showed that the hypothesis is accurate except for very small time separations, where small but measurable differences between computed and measured correlations exist. On theoretical grounds, Kraichnan (1957) concluded that the hypothesis was inconsistent with the equations of motion. It is of interest to further test the hypothesis (in an extended form) for correlations involving more than two-points. To minimize computation time, we have determined this correlation only for certain restricted values of the separations, as follows. The values were chosen such that the transforms available from the triple correlation computations (stored on magnetic tape) could be used again, thus avoiding recomputation of some transforms.

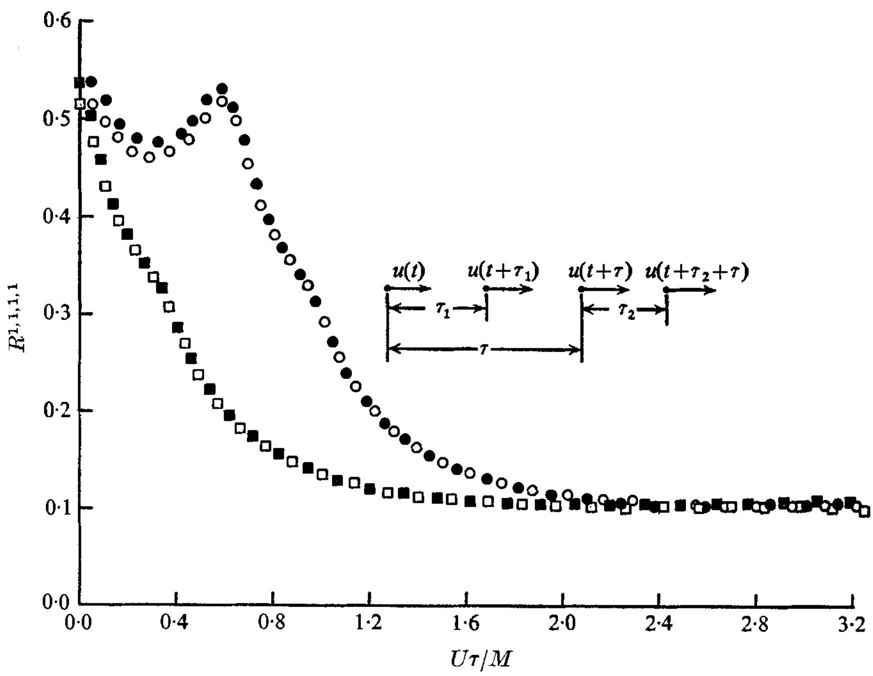

FIGURE 5. Fourth-order correlations.

$$R^{1,1,1,1}(\tau_1,\tau,\tau_2+\tau) = \langle u(t)u(t+\tau_1)u(t+\tau)u(t+\tau_2+\tau)\rangle/\langle u^2\rangle^2$$

for fixed $U\tau_1/M = 0{\cdot}94464$ and $U\tau_2/M = 0{\cdot}32388$. (*a*) Measured correlations: ●, positive τ; ■, negative τ. (*b*) Calculated from Gram-Charlier third-order non-Gaussian probability distribution: ○, positive τ; □, negative τ.

For example,
$$S_{1,1}(\tau_1,\tau_2,f) = (1/T)[U_1(\tau_2,f)U_1^*(\tau_1,f)]$$

could be immediately determined since $U_1(\tau_1,f)$ were available from the triple correlation computations, and then

$$R^{1,1,1,1}(\tau_1,\tau,\tau+\tau_2)\langle u^2\rangle^2 = \int_{-\infty}^{\infty} S_{1,1}(\tau_1,\tau_2,f)\exp(i2\pi f\tau)\,df$$
$$= \langle u(t)u(t+\tau_1)u(t+\tau)u(t+\tau_2+\tau)\rangle$$

was determined from a single transform. Similarly, since

$$u_2(\tau_1,\tau_2,t) = u(t)u(t+\tau_1)u(t+\tau_2)$$

and
$$U_2(\tau_1,\tau_2,f) = \int_{-\infty}^{\infty} u_2(\tau_1,\tau_2,t)\exp(-2\pi ft)\,dt.$$

Then
$$S_{2,0}(\tau_1,\tau_2,f) = (1/T)[U(f)U_2^*(\tau_1,\tau_2,f)]$$

and the correlation is

$$R^{1,1,1,1}(\tau_1, \tau_2, \tau)\langle u^2\rangle^2 = \int_{-\infty}^{\infty} S_{2,0}(\tau_1, \tau_2, f)\exp(i2\pi f\tau)\,df$$
$$= \langle u(t)\,u(t+\tau_1)\,u(t+\tau_2)\,u(t+\tau)\rangle.$$

These particular four-point correlations are two cross-sectional cuts through the general correlation function. The first is easily computed with minimal computer time if the $U_1(\tau, f)$ are stored on digital tape, while for the second one we first form the additional function $U_2(\tau_1, \tau_2, f)$. The values of the time separations chosen were the same as used for the triple correlations.

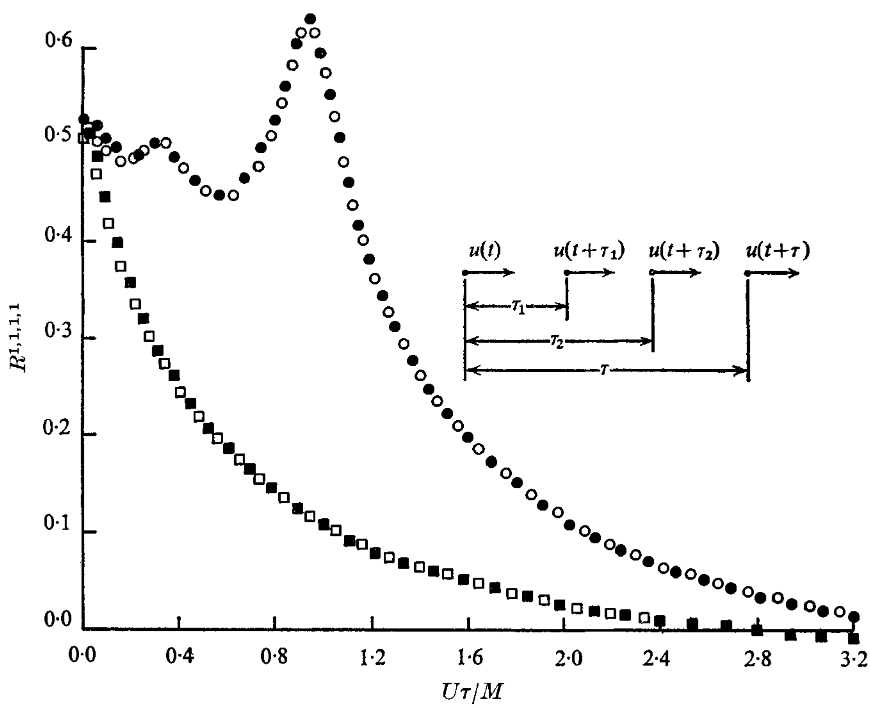

FIGURE 6. Fourth-order correlations. $R^{1,1,1,1}(\tau_1, \tau_2, \tau) = \langle u(t)u(t+\tau_1)u(t+\tau_2)u(t+\tau)\rangle/\langle u^2\rangle^2$ for fixed $U\tau_1/M = 0{\cdot}32388$ and $U\tau_2/M = 0{\cdot}94464$. (*a*) Measured correlations: ●, positive τ; ■, negative τ. (*b*) Calculated from Gram-Charlier third-order non-Gaussian probability distribution: ○, positive τ; □, negative τ.

The correlations for both positive and negative τ are shown in figures 5 and 6. The correlations are very smooth, although less data were used than for the triple correlations. This relative behaviour is very similar to that found for the even- and odd-order two-point correlations in I, and reflects the fact that even-point correlations are dominated by the nearly Gaussian nature of the signal, while the odd-point correlations are a direct measure of the small departures from Gaussian statistics. These correlations are not symmetrical about $\tau = 0$ since the cuts they represent are not directly across planes of symmetry. However, $R_4(\tau_1, \tau, \tau_2+\tau)$ is taken parallel to a plane of symmetry and is symmetric about the point $\tau = 0{\cdot}31$. Also shown in figures 5 and 6 are the corresponding four-point correlations computed from the two-point correlation using Millionshtchikov's hypothesis, i.e.

$$R^{1,1,1,1} = R_{1,2}R_{3,4} + R_{1,3}R_{2,4} + R_{1,4}R_{2,3}.$$

These results are in excellent agreement with the measured correlations, except for a small region near $|\bar{\tau}| = 0$, $(\bar{\tau} = (\tau_1^2 + \tau_2^2 + \tau^2)^{\frac{1}{2}})$ where the differences increase to about 4 %. Letting $dr = Vdt$ by Taylor's hypothesis and interpreting the results as space correlations, we infer from the present measurements and those of I that some of the difficulties arising in theoretical investigations employing the Millionshtchikov hypothesis are due to its failure for the smallest spatial separations.

With respect to the differences near $|\tau| = 0$, we note that in a similar situation Frenkiel & Klebanoff (1967 *a*) found that the Gram-Charlier distribution provided an improved fit for the even-order two-point correlations for small separations. However, in the present case, the appropriate third-order, four-point Gram-Charlier probability distribution yields precisely the same relation between four- and two-point correlations as the joint-Gaussian hypothesis, and hence no improved fit to the data is obtained from the Gram-Charlier approximation in this case.

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